COMPACT OPERATORS AND THE ORLICZ-PETTIS PROPERTY IN p-ADIC ANALYSIS

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C. Perez-Garcia and W.H. Schikhof

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DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
Toernooiveld
6525 ED Nijmegen
The Netherlands

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ABSTRACT. For a non-archimedean locally convex space E we study in this paper the property

1. "Every weakly convergent sequence in E is convergent"

as related to

2. "Every continuous linear map from ℓ^∞ to E is compact".

Also, we show that for a Banach space E there is a duality between these properties and the property

3. "Every $\sigma(E', E)$-convergent sequence in E' is norm convergent in E'",

which has been studied by N. de Grande-de Kimpe in [1] and [2] and recently by T. Kiyosawa in [4].

TERMINOLOGY

Throughout K is a non-archimedean valued field that is complete under the metric induced by the non-trivial valuation $|\cdot|$, and $E, F, \ldots$ are locally convex spaces over K. We always assume that $E, F \ldots$ are Hausdorff.

$L(E, F)$ will be the K-vector space consisting of all continuous linear maps $E \to F$. The topological dual space of E is $E' := L(E, K)$. Also, the algebraic dual space of E will be denoted E^*. Observe that the weak topology $\sigma(E, E')$ of E is Hausdorff if and only if E' separates the points of E.

By $E \simeq F$ we mean that there is a linear homeomorphism from E onto F.

A nonempty subset A of E is *absolutely convex* if $x, y \in A$, $\lambda, \mu \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$ implies $\lambda x + \mu y \in A$. The absolutely convex hull of A is denoted by $co(A)$ and the K-linear hull of A by $[A]$. We shall write $\overline{co}(A)$ instead of $\overline{co(A)}$.

If p is a continuous seminorm in E we denote by E_p the associated normed space $E/\text{Ker } p$.

For unexplained terms and background we refer to [6] (locally convex spaces) and [13] (normed spaces).

§1. (O.P.) - SPACES.

The classical Banach space ℓ^1 (over $\mathbf{R}$ or $\mathbf{C}$) has the property that every weakly convergent sequence is norm convergent. This fact is known as the Orlicz - Pettis Theorem. Let us define in the non-archimedean theory:

DEFINITION 1.1. A locally convex space E over K is called an *Orlicz - Pettis space* ((O.P.)-*space*) if every weakly convergent sequence is convergent (or equivalently, if for every sequence $x_1, x_2, \ldots$ in E, $\lim_{n\to\infty} x_n = 0$ weakly implies $\lim_{n\to\infty} x_n = 0$ in the original topology).

It is well-known that c_o is an (O.P.)-space ([13], p. 158).

Also, it follows easily that if E is an (O.P.)-space, then the dual E' separates the points of E. However, ℓ^∞ is a space whose dual separates the points, but, if K is not spherically complete, is not an (O.P.)-space (see [6], Remark following Proposition 4.11).

We now study some stability properties of the class of (O.P.)-spaces.

PROPOSITION 1.2.

a) *A subspace of an* (O.P.)-*space is an* (O.P.)-*space.*

b) *The product of a family of* (O.P.)-*spaces is an* (O.P.)-*space.*

Proof.

a) Let D be a subspace of the (O.P.)-space (E, τ) and let $x_1, x_2, \ldots$ be a sequence in D converging weakly to 0 in $\sigma(D, D')$. Then, certainly $x_n \to 0$ with respect to $\sigma(E, E')$. Hence, $x_n \xrightarrow{\tau} 0$ by assumption.

b) Let, for each i belonging to some set I, E_i be an (O.P.)-space and let $(x_i^1)_{i\in I}$, $(x_i^2)_{i\in I}, \ldots$ converge weakly to 0 in the product space $\prod_{i\in I} E_i$. By continuity of projections $x_i^1, , x_i^2, \ldots$ converges weakly to 0 in E_i for each $i \in I$, hence by assumption in the initial topology of E_i. But this is precisely convergence to 0 of $(x_i^1)_{i\in I}, (x_i^2)_{i\in I}, \ldots$ in the product topology.

Let us denote the locally convex direct sum of a family $\{E_i : i \in I\}$ of locally convex spaces by $\bigoplus_{i\in I} E_i$. Recall that each $e \in \bigoplus_{i\in I} E_i$ has a unique decomposition $e = \sum_{i\in I} e_i$, where $e_i \in E_i$ for each i and where $\{i \in I : e_i \neq 0\}$ is finite.

LEMMA 1.3. *Let $\{E_i : i \in I\}$ be a family of locally convex spaces such that the weak topology on each E_i is Hausdorff. Then, for any weakly bounded set X in $\bigoplus_{i\in I} E_i$,*

$$J := \{i \in I : \text{ there exists an } x \in X \text{ with } x_i \neq 0\}$$

is finite.

Proof. Suppose J is infinite. Then inductively one can find $i_1, i_2, \ldots \in I$ and $x^1, x^2, \ldots \in X$ such that $x^j_{i_m} = 0$ if $j < m$ and $x^m_{i_m} \neq 0$ for each $m \in \{1, 2, \ldots\}$. Then, again inductively, one can construct $f_{i_1} \in E'_{i_1}, f_{i_2} \in E'_{i_2}, \ldots$ such that for each $m \in \mathbb{N}$

$$|f_{i_m}(x^m_{i_m})| \geq m + |\sum_{k<m} f_{i_k}(x^m_{i_k})|.$$

If $i \in I \backslash \{i_1, i_2, \ldots\}$ we define $f_i \in E'_i$ to be 0. Then, the formula

$$f(\sum_{i \in I} e_i) = \sum_{i \in I} f_i(e_i)$$

defines an element $f \in (\bigoplus_{i \in I} E_i)'$. Also for each $m \in \mathbb{N}$ we have

$$|f(x^m)| = |\sum_{i \in I} f_i(x^m_i)| = |\sum_{k \in \mathbb{N}} f_{i_k}(x^m_{i_k})| =$$
$$= |\sum_{k \leq m} f_{i_k}(x^m_{i_k})| \geq |f_{i_m}(x^m_{i_m}| - |\sum_{k<m} f_{i_k}(x^m_{i_k})| \geq m.$$

It follows that X is not weakly bounded, a contradiction

PROPOSITION 1.4. *The locally convex direct sum of a family of* (O.P.)*-spaces is again an* (O.P.)*-space.*

Proof. Any weakly convergent sequence in the direct sum $\bigoplus_{i \in I} E_i$ of the (O.P.)-spaces E_i is, by weak boundedness and Lemma 1.3, contained in $\bigoplus_{i \in J} E_i \simeq \prod_{i \in J} E_i$ for some finite set $J \subset I$, and also weakly convergent in that space, hence, by Proposition 1.2.b), convergent in the restricted topology of $\bigoplus_{i \in J} E_i \subset \bigoplus_{i \in I} E_i$.

REMARK. The class of (O.P.)-spaces is not closed for forming of quotients.

Indeed, let K be not spherically complete. Then ℓ^∞ is not an (O.P.)-space. On the other hand, one can make a quotient map $c_0(I) \to \ell^\infty$ if I has sufficiently large cardinal, and we shall see in Theorem 1.6 (vi) that $c_0(I)$ is an (O.P.)-space.

However, we do have the following

PROPOSITION 1.5. *Let E be a locally convex space and let D be a finite dimensional subspace.*

(i) *If E is an* (O.P.)*-space then so is E/D.*

(ii) *If E' separates the points of E and if E/D is an* (O.P.)*-space, then so is E.*

Proof. An elementary reasoning shows that, if E' separates the points of E, then D is complemented in E, i.e., $E = D \oplus H$ for some closed subspace H of E. It follows that H is linearly homeomorphic to E/D. Now apply Proposition 1.2.a) to find (i) and Proposition 1.4 to find (ii).

To obtain examples of (O.P.)-space (in Theorem 1.6) we first recall some definitions.

Following [6] a locally convex space E is *of countable type* if for every continuous seminorm p the normed space E_p is of countable type (a normed space is *of countable type* if there exists a countable set whose linear span is dense). Also, E is *strongly polar* if for every continuous seminorm p the formula $p = \sup\{|f| : f \in E^*, |f| \leq p\}$ holds. Finally, following [12] we say that E *has property* $(*)$ if for each subspace D of countable type, each $f \in D'$ has a continuous linear extension $\overline{f} \in E'$.

THEOREM 1.6. *The following spaces are* (O.P.)*-spaces.*

(i) *Any locally convex space E such that for every continuous seminorm p on E, the associated normed space E_p is an* (O.P.)*-space.*
(ii) *Every locally convex space of countable type.*
(iii) *Every locally convex space with the property* $(*)$.
(iv) *Every strongly polar space.*
(v) *Every locally convex space over a spherically complete field.*
(vi) *Every Banach space with a base.*
(vii) *Any vector space E equipped with the strongest locally convex topology τ^*.*

Proof.
(i) We know that if $\mathcal{P}$ is a family of seminorms determining the topology of E, then E can be considered as a subspace of $\prod_{p\in\mathcal{P}} E_p$. Now apply Proposition 1.2.
Property (ii) is a direct consequence of property (i).
Also, the proof of (iii) is just like the corresponding one given in [12], Theorem 5.2 for normed spaces.
Now (iv), (v) and (vi) are special cases of (iii) (see [6], Theorem 4.2 and [13], Corollary 3.18).
To prove (vii) just observe that (E, τ^*) is linearly homeomorphic to $\bigoplus_{i\in I} K_i$ where I has the cardinality of an algebraic base of E and where $K_i = K$ for each i. Now apply Proposition 1.4.

§2. (O.P.)-LIKE SPACES.

In this section we consider the following variants of the Orlicz-Pettis property.

DEFINITION 2.1. Let E be a locally convex space over K.
(a) E is called a (B.O.P.)-*space* if every bounded weakly convergent sequence is convergent.
(b) E is called a (C.O.P.)-*space* if every Cauchy sequence that is weakly convergent, is convergent.
Clearly we have

E is an (O.P.)-*space* $\Rightarrow$ *E is a* (B.O.P.)-*space* $\Rightarrow$ *E is a* (C.O.P.)-*space*

and also that the dual of a (C.O.P.)-space separates points.

Further, it is not hard to see by looking at the proofs of 1.2 - 1.5 that the class of (B.O.P.)-spaces ((C.O.P.)-spaces) is stable with respect to subspaces, products, locally convex direct sums and quotients by finite dimensional subspaces.

The space ℓ^∞, over a nonspherically complete K, is a (C.O.P.)-space but, as we saw in §1, not a (B.O.P.)-space.

If E is a normed space, we have that *E is a* (B.O.P.)-*space* $\Longleftrightarrow$ *E is an* (O.P.)-*space*. Indeed, suppose that E is a (B.O.P.)-space and that $x_1, x_2, \ldots$ tends to zero weakly but $\|x_n\| \uparrow \infty$. Then there exist $\lambda_1, \lambda_2, \ldots \in K$ such that $\{\|\lambda_n x_n\| : n \in \{1,2,\ldots\}\}$ is bounded away from 0 and $\lambda_n \to 0$. Then $\lambda_1 x_1, \lambda_2 x_2, \ldots$ is a bounded sequence such that $\lambda_n x_n \to 0$ weakly. Hence, $\|\lambda_n x_n\| \to 0$, a contradiction.

From this fact, the following question arises in a natural way:

PROBLEM: Is every (B.O.P.)-space an (O.P.)-space?

In 2.1, 2.2, 2.3 below we shall prove characterizations of (C.O.P.)-, (B.O.P.)-, (O.P.)-spaces respectively, yielding a comparison between these classes. Recall ([3]) that an absolutely convex subset A of a locally convex space E is said to be (a) *compactoid* if for each neighbourhood U of 0 in E there exists a finite set $H \subset E$ such that $A \subset U + co(H)$.

Then,

THEOREM 2.2. *For a locally convex space E, the following properties are equivalent.*
(i) *E is a* (C.O.P.)-*space.*
(ii) *On metrizable and compactoid sets in E, the weak topology $\sigma(E, E')$ and the original topology τ coincide.*
(iii) *For every metrizable and compactoid set $A \subset E$, $\overline{A}^{\sigma(E,E')} = \overline{A}^{\tau}$.*
(iv) *Every closed metrizable and compactoid subset of E is weakly closed.*

Proof.

(i) $\Rightarrow$ (ii). Let $A \subset E$ be a metrizable and compactoid subset of E and let $\hat{E}$ denote the completion of E. Set

$$E^s := \{x \in \hat{E}\colon \text{ there is a sequence } (x_n) \text{ in } E \text{ such that } x_n \to x \text{ in } \hat{E}\}$$

By [8], Theorem 6.1 we have that $\overline{A}^{E^s}$ is also a metrizable and compactoid subset of E^s endowed with the restricted topology induced by $\hat{E}$. Also, $\overline{A}^{E^s}$ is complete.

On the other hand, one proves very easily that if E is a (C.O.P.)-space then the weak topology $\sigma(E^s, (E^s)')$ is Hausdorff.

Now apply Theorem 3.2 of [7] to conclude that $\sigma(E, E')|A = \tau|A$.

The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious.

Finally we prove (iv) $\Rightarrow$ (i). Let (x_n) be a Cauchy sequence converging weakly to 0. By [8], Theorem 6.1 we have that $A := \overline{co}(x_1, x_2, \ldots) = \overline{co}(x_1, x_1 - x_2, x_2 - x_3, \ldots)$ is a metrizable compactoid subset of E.

Metrizability of A implies the existence of a sequence $U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$ of clopen neighbourhoods of 0 such that $\{U_n \cap A : n \in \{1, 2, \ldots\}\}$ generate $\tau|A$.

Take $m \in \{1, 2, \ldots\}$. Since (x_n) is a Cauchy sequence we have that $x_k - x_\ell \in U_m \cap A$ for sufficiently large k, ℓ. Also, $U_m \cap A$ is a closed metrizable compactoid set and by (iv) it is also weakly closed. Further, (x_ℓ) converges weakly to 0. Putting together these facts we conclude that $x_k \in U_m \cap A$ for large k. Thus, $x_1, x_2, \ldots$ converges to 0 in the original topology τ, and we are done.

COROLLARY 2.3. *For a locally convex space E, the following are equivalent.*

(i) *E is a (B.O.P.)-space.*

(ii) *E is a (C.O.P.)-space and every absolutely convex bounded and $\sigma(E, E')$-metrizable set of E is metrizable and compactoid.*

(iii) *E' separates the points of E and on every absolutely convex bounded and $\sigma(E, E')$-metrizable set in E, the weak topology $\sigma(E, E')$ and the original topology τ coincide.*

Proof.

(i) $\Rightarrow$ (ii). We already know that if E is a (B.O.P.)-space then E is a (C.O.P.)-space and E' separates the points. Now let $A \subset E$ be an absolutely convex bounded and $\sigma(E, E')$-metrizable subset of E. Let $\lambda \in K, |\lambda| > 1$ if the valuation is dense, $\lambda = 1$ if the valuation is discrete. By [6], Proposition 8.2 there exists a sequence $e_1, e_2, \ldots$ in λA (and hence it is a bounded sequence) with $\lim_{n\to\infty} e_n = 0$ in $\sigma(E, E')$ and such that $A \subset \overline{co}^{\sigma(E,E')}(e_1, e_2, \ldots)$. Since E is a (B.O.P.)-space we conclude that $e_n \xrightarrow{\tau} 0$, which implies that $co(e_1, e_2, \ldots)$ is a metrizable compactoid set ([8], Theorem 6.1). By

Theorem 2.2 it follows that $\overline{co}^{\sigma(E,E')}(e_1, e_2, \ldots)$ (and hence also A) is a metrizable and compactoid set.

The implication (ii) $\Rightarrow$ (iii) is a direct consequence of Theorem 2.2.

(iii) $\Rightarrow$ (i). Let (x_n) be a bounded sequence such that $x_n \to 0$ weakly. Then, $A := co(x_1, x_2, x_3, \ldots)$ is a bounded and $\sigma(E, E')$-metrizable subset of E ([8], Theorem 6.1). By (iii), $\tau|A = \sigma(E, E')|A$. Hence, $x_n \xrightarrow{\tau} 0$.

COROLLARY 2.4. *For a locally convex space E the following are equivalent.*

(i) *E is an* (O.P.)*-space.*

(ii) *E is a* (B.O.P.)*-space and every weakly bounded set is bounded.*

(iii) *E is a* (C.O.P.)*-space and every absolutely convex weakly bounded and $\sigma(E, E')$-metrizable subset of E is metrizable and compactoid.*

(iv) *E' separates the points of E and for every absolutely convex weakly bounded and $\sigma(E, E')$-metrizable set in E, the weak topology $\sigma(E, E')$ and the original topology coincide.*

Proof.

(i) $\Rightarrow$ (ii). We only have to prove that if E is an (O.P.)-space then every weakly bounded set is bounded. Thus, let $X \subset E$ be weakly bounded but not τ-bounded. Then, there is a τ-continuous seminorm p and a sequence $x_1, x_2, \ldots$ in X such that $p(x_n) \geq n$ for each n. There is a $\rho > 0$ and there are $\lambda_1, \lambda_2, \ldots$ in K such that $\rho \leq p(\lambda_n x_n) \leq 1$ for each n. Then $\lambda_n \to 0$ and, by the weak boundedness of $\{x_1, x_2, \ldots\}$, $\lim_{n\to\infty} \lambda_n x_n = 0$ weakly. By (i), $\lambda_n x_n \xrightarrow{\tau} 0$ which is impossible as $p(\lambda_n x_n) \geq \rho$ for all n.

The implication (ii) $\Rightarrow$ (iii) follows directly from Corollary 2.3 (i) $\Rightarrow$ (ii) and the implication (iii) $\Rightarrow$ (iv) follows from Theorem 2.2 (i) $\Rightarrow$ (ii). Finally the implication (iv) $\Rightarrow$ (i) can be proved just like the corresponding one in Corollary 2.3.

In the next section we shall study the (O.P.)-property from a different angle. With an eye on Theorem 1.6 (v) we shall assume that K is not spherically complete.

§3. (∞)-SPACES.

FROM NOW ON IN THIS PAPER WE ASSUME THAT K IS NOT SPHERICALLY COMPLETE

DEFINITION 3.1. A locally convex space E is said to be an (∞)-*space* if every continuous linear map $\ell^\infty \to E$ is compact (i.e. the image of the unit ball of ℓ^∞ is a compactoid set in E).

The next theorems reveal the connection with the previous sections.

THEOREM 3.2. *For a locally convex space (E, τ) consider the following properties,*
(α) E is an (O.P)-space.
(β) E is an (∞)-space and every weakly bounded set is bounded.
(γ) E is an (∞)-space, E' separates the points of E and every absolutely convex weakly bounded and $\sigma(E,E')$-metrizable set is bounded.
Then, $(\alpha) \Rightarrow (\beta) \iff (\gamma)$.
If in addition E is weakly sequentally complete, then properties $(\alpha) - (\gamma)$ are equivalent.

Proof.
$(\alpha) \Rightarrow (\beta)$. It suffices, by Corollary 2.4 (i) $\Rightarrow$ (ii), to show that E is an (∞)-space. So let $T \in L(\ell^\infty, E)$ and let $e_1 := (1,0,0,\ldots)$, $e_2 = (0,1,0,\ldots), \ldots$ be the unit vectors of ℓ^∞. K is not spherically complete so $(\ell^\infty)' \simeq c_o$ ([13], Theorem 4.17) and therefore for each $y = (\eta_1, \eta_2, \ldots)$ in ℓ^∞ we have $y = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \eta_i e_i$ weakly. By weak continuity

$$Ty = \lim_{n\to\infty} \sum_{i=1}^{n} \eta_i T e_i \text{ weakly.}$$

By (α) the above limit is in the sense of τ. Thus, the unit ball of ℓ^∞ is mapped into the τ-closure of $co\{Te_1, Te_2, \ldots\}$ which is a compactoid since $Te_n \xrightarrow{\tau} 0$. We conclude that T is compact and that E is an (∞)-space.
The proof of $(\beta) \Rightarrow (\gamma)$ is elementary; we prove $(\gamma) \Rightarrow (\beta)$. Suppose $X \subset E$ is weakly bounded but not bounded. Then there exist a continuous seminorm p and a sequence $x_1, x_2, \ldots$ in X such that $p(x_n) \to \infty$. There exist $\lambda_1, \lambda_2, \ldots \in K$ such that $\lim\limits_{n\to\infty} \lambda_n = 0$ and $p(\lambda_n x_n) \to \infty$. As $\lambda_n \to 0$ and $\{x_1, x_2, \ldots\}$ is weakly bounded we have $\lambda_n x_n \to 0$ weakly. Then
$co(\lambda_1 x_1, \lambda_2 x_2, \ldots)$ is weakly metrizable, weakly bounded, absolutely convex, hence bounded by (γ). But this conflicts $p(\lambda_n x_n) \to \infty$.

Now assume that E is weakly sequentally complete.

$(\gamma) \Rightarrow (\alpha)$. Let $x_1, x_2, \ldots$ be a sequence in E converging weakly to 0. Then the formula

$$(\eta_1, \eta_2, \ldots) \stackrel{T}{\longmapsto} \sigma(E, E') - \lim_{n\to\infty} \sum_{i=1}^{n} \eta_i x_i$$

defines, by weak sequential completeness, a linear map $T : \ell^\infty \to E$. It maps the unit ball of ℓ^∞ into the set $A := \overline{co}^{\sigma(E,E')}(x_1, x_2, \ldots)$. Since $x_n \to 0$ weakly we have that A is a weakly bounded and $\sigma(E, E')$-metrizable set and from (γ) we derive that A is bounded. Then T maps the unit ball into a bounded set so that T is continuous. By assumption T is compact. Then $T(\ell^\infty)$ is a space of countable type ([7], Proposition 4.3). We have that $x_n = Te_n \to 0$ in the topology $\sigma(T\ell^\infty, (T\ell^\infty)')$. Theorem 1.6 implies that $x_n \xrightarrow{\tau} 0$.

THEOREM 3.3. *For a locally convex space (E, τ) consider the following properties.*
(α) *E is a* (B.O.P.)*-space.*
(β) *E is an (∞)-space, E' separates the points of E and for every absolutely convex bounded and $\sigma(E, E')$-metrizable set $A \subset E$, $\overline{A}^{\sigma(E,E')}$ is bounded.*
Then, $(\alpha) \Rightarrow (\beta)$. If in addition E is weakly sequentally complete, then also $(\beta) \Rightarrow (\alpha)$.

Proof.
$(\alpha) \Rightarrow (\beta)$. Observe that for each $T \in L(\ell^\infty, E)$ and for each $y := (\eta_1, \eta_2, \ldots)$ in ℓ^∞, the sequence consisting of the partial sums $\sum_{i=1}^{n} \eta_i e_i$ is bounded. As in $(\alpha) \Rightarrow (\beta)$ of Theorem 3.2 we conclude that E is an (∞)-space. The rest follows from Corollary 2.3.

Now, assume that E is weakly sequentally complete. Then $(\beta) \Rightarrow (\alpha)$ can be proved in a similar way as $(\gamma) \Rightarrow (\alpha)$ in Theorem 3.2.

REMARKS.

1. There exist (C.O.P.)-spaces which are not (∞)-spaces (e.g. ℓ^∞).
2. Of course, $(\alpha) \Rightarrow (\beta)$ in Theorems 3.2 and 3.3 is not true if the field is spherically complete.
3. There exist Banach (∞)-spaces which are not (B.O.P.)-spaces. To construct an example we need a preliminary concept. Let us say that an *ultrametric group* is an abelian group G, together with an invariant ultrametric d. A surjective homomorphism $\varphi : (G_1, d_1) \to (G_2, d_2)$, where $(G_1, d_1), (G_2, d_2)$ are ultrametric groups, is called a *quotient map* if

$$d_2(\varphi(x), 0) = \inf\{d_1(y, 0) : y \in G_1, \varphi(x) = \varphi(y)\}$$

for each $x \in G_1$.

The following result is crucial for our purpose.

THEOREM 3.4. *Let E, F be Banach spaces over K. Let $\varphi : E \to F$ be a quotient map. Let $D :=$ Ker φ. Then, if F is spherically complete we have for each $f \in E'$,*

$$\sup_{x \in B_E} |f(x)| = \sup_{x \in B_D} |f(x)| \tag{I}$$

(where B_E and B_D denote the "closed" unit ball in E and D respectively).

Proof. Suppose there exists an $f \in E'$ such that $\|f\| = 1$ and $|f| \le r$ on B_D for some $r < 1$. Let $\rho : B_K \to B_K/\{\lambda \in K : |\lambda| \le r\} =: k_r$ be the canonical surjection. With the natural quotient metric k_r is an ultrametric group (which is easily seen to be not spherically complete) and ρ is a quotient map in the sense of above. Then $\rho \circ f : B_E \to k_r$ is a quotient map as well, which is zero on B_D. The restriction of φ to B_E (again called φ) is a quotient map $\varphi : B_E \to \varphi(B_E)$ whose kernel equals B_D.

There is a unique homomorphism of groups ψ making

$$\begin{tikzcd} B_E \arrow[rr, "\varphi"] \arrow[dr, "\rho \circ f"'] & & \varphi(B_E) \arrow[dl, "\psi"] \\ & k_r & \end{tikzcd}$$

commute. Since φ and $\rho \circ f$ are quotient maps, so is ψ. Also, as φ is a quotient map, $\varphi(B_E)$ is spherically complete. Similarly, k_r is spherically complete, a contradiction.

Now we are ready to provide examples of (∞)-spaces which are not (O.P.)-spaces.

THEOREM 3.5. *Let E be an (∞)-Banach space and suppose there is a closed subspace $D \neq E$ satisfying property* (I) *of Theorem* 3.4. (For example, take $E := c_o(X)$ where X is the "closed" unit ball of $F := \ell^\infty/c_o$, $D :=$ Ker φ where $\varphi : E \to F$ is given by $\varphi(\Sigma\lambda_x e_x) = \Sigma\lambda_x x$; now apply Theorem 3.4.)

For each $n \in \mathbb{N}$, let E_n be the space E endowed with the norm

$$N_n(x) = \max(\|x\|, n \text{ dist } (x, D))$$

Then

$$G := \{(x_1, x_2, \ldots) \in \prod_{n \in \mathbb{N}} E_n : \lim_{n \to \infty} N_n(x_n) = 0\}$$

with the norm given by

$$N(x_1, x_2, \ldots) = \max_n N_n(x_n)$$

is an (∞)-Banach space which is not an (O.P.)*-space.*

Proof. Firstly observe that the norms N_n and $\|\cdot\|$ are equivalent so that, as sets, $E'_n = E'$ for each $n \in \mathbf{N}$. Also, by property (I) it follows directly that the identity map from E' onto E'_n is an isometry.

Now we prove that G is an (∞)-space. Let $T \in L(\ell^\infty, G)$ and for each $n \in \mathbf{N}$ let $\pi_n : G \to E_n$ be the obvious continuous projection. Then $\pi_n \circ T \in L(\ell^\infty, E_n) = L(\ell^\infty, E)$ is a compact map, so $\pi_n T(\ell^\infty)$ is of countable type. Then $T\ell^\infty$ is in the N-closure of $\sum_n \pi_n T(\ell^\infty)$ and is therefore of countable type. Now, apply Theorem 3.9 $(\alpha) \iff (\gamma)$ below.

Finally we show that (G, N) is not an (O.P.)-space. For each $n \in \mathbf{N}$ choose $u_n \in E_n$ such that $\|u_n\| \leq \frac{1}{\sqrt[4]{n}}$, $\operatorname{dist}(u_n, D) \geq \frac{1}{\sqrt{n}}$ and set

$$z_n = (0, 0, \ldots, u_n, 0, \ldots) \in G.$$

We see that $N(z_n) = N_n(u_n) \geq n \operatorname{dist}(u_n, D) \geq \sqrt{n}$, so the sequence $\{z_1, z_2, \ldots\}$ is not bounded in (G, N). On the other hand, let $f \in (G, N)'$. Then (see [13], Exercise 3.Q) there exist $f_n \in E'_n$ such that

$$f((x_1, x_2, \ldots)) = \sum_{n=1}^{\infty} f_n(x_n) \qquad ((x_1, x_2, \ldots) \in G)$$

while

$$M := \sup_n N_n(f_n) < \infty.$$

We have:

$$|f(z_n)| = |f_n(u_n)| \leq \|f_n\| \, \|u_n\| = N_n(f_n)\|u_n\| \leq M\|u_n\| \leq M/\sqrt[4]{n}.$$

so $z_n \to 0$ weakly. Thus, (G, N) is not an (O.P.)-space.

Next, we shall see that for (∞)-spaces we have the same stability properties as for (O.P.) ((B.O.P.) or (C.O.P.))-spaces. To see that we need the following lemma.

LEMMA 3.6. *Let $\{E_i : i \in I\}$ be a family of locally convex spaces. Then for any bounded set $X \subset \bigoplus_{i \in I} E_i$*

$$J := \{i \in I : \text{ there is an } x \in X \text{ with } x_i \neq 0\}$$

is a finite set.

Proof. Suppose we have distinct $i_1, i_2, \ldots \in I$ such that for each n there exists an $x^n \in X$ with $x^n_{i_n} \neq 0$. Then there are continuous seminorms q_{i_1} on E_{i_1}, q_{i_2} on $E_{i_2}, \ldots$ such that

$q_{i_n}(x^n_{i_n}) \geq n$ for each n. For $i \in I \backslash \{i_1, i_2, \dots\}$ we define q_i to be zero on E_i. Then $q : z \mapsto \max_{i \in I} q_i(z_i)$ is a continuous seminorm on $\bigoplus_{i \in I} E_i$. We have $q(x^n) \geq q_{i_n}(x^n_{i_n}) \geq n$ for each n so that X is not bounded, a contradiction.

PROPOSITION 3.7.

a) *A subspace of an (∞)-space is again an (∞)-space.*

b) *The product of a family of (∞)-spaces is again an (∞)-space.*

c) *The locally convex direct sum of a family of (∞)-spaces is again an (∞)-space.*

d) *If E is a locally convex space such that E' separates the points of E and D is a finite dimensional subspace of E, then*

d.i) *If E is an (∞)-space then so is E/D.*

d.ii) *If E/D is an (∞)-space then so is E.*

Proof. The proof of a) is direct verification and the proof of (d) is analogous to the proof of Proposition 1.5.

b) Let $\{E_i : i \in I\}$ be a family of (∞)-spaces and let $T \in L(\ell^\infty, \prod_{i \in I} E_i)$. Then $\pi_i \circ T$ (where π_i is the projection onto E_i) is compact for each i, implying that $TB_{\ell^\infty} \subset \prod_{i \in I} \pi_i(TB_{\ell^\infty})$ (where B_{ℓ^∞} denotes the (closed) unit ball of ℓ^∞) is a compactoid, i.e., T is compact.

c) Let $\{E_i : i \in I\}$ be a family of (∞)-spaces and let $T \in L(\ell^\infty, \bigoplus_{i \in I} E_i)$. Then TB_{ℓ^∞} is bounded so it lies in $\bigoplus_{i \in J} E_i$ for some finite $J \subset I$ by Lemma 3.6. Now apply b) and the fact that $\bigoplus_{i \in J} E_i \simeq \prod_{i \in J} E_i$ to arrive at the compactness of T.

It turns out that we can sharpen statement d.ii) for (∞)-spaces (see Proposition 3.11). As a stepping stone we prove several characterizations of (∞)-spaces (see Theorem 3.9).

LEMMA 3.8. *Let (E, τ) be a locally convex space and let $T \in L(\ell^\infty, E)$. Then T is compact if and only if it has the form*

$$(\eta_1, \eta_2, \dots) \mapsto \sum_{n=1}^{\infty} \eta_n x_n \qquad (*)$$

where $x_1, x_2, \dots \in E$, $x_n \xrightarrow{\tau} 0$. In particular, if T is compact then TB_{ℓ^∞} is a metrizable compactoid.

Proof. Suppose T is compact. Set $x_n := Te_n$ where $e_1, e_2, \dots$ are the unit vectors of ℓ^∞. Then $e_n \to 0$ weakly so $x_n \to 0$ weakly in $F = T\ell^\infty$, a space of countable type, hence an (O.P.)-space. So $x_n \xrightarrow{\tau} 0$ and

$$T(\eta_1, \eta_2, \ldots) = \sigma(E, E') - \lim_{n\to\infty} \sum_{i=1}^{n} \eta_i x_i =$$

$$= \tau - \lim_{n\to\infty} \sum_{i=1}^{n} \eta_i x_i = \sum_{i=1}^{\infty} \eta_i x_i$$

for each $(\eta_1, \eta_2, \ldots) \in \ell^\infty$.

Conversely, if T has the form $(*)$ then TB_{ℓ^∞} is contained in the τ-closure of $co\{x_1, x_2, \ldots\}$ where $x_n \xrightarrow{\tau} 0$. So TB_{ℓ^∞} is a τ-metrizable (see [8], Theorem 6.1) compactoid.

THEOREM 3.9. *For a locally convex space E the following are equivalent:*
(α) *E is an (∞)-space.*
(β) *For every $T \in L(\ell^\infty, E)$ the image of the unit ball is a metrizable compactoid.*
(γ) *For every $T \in L(\ell^\infty, E)$ the range of T is of countable type.*
(δ) *For every $T \in L(\ell^\infty, E)$ the range of T is an* (O.P.)*-space.*
(ε) *For every $T \in L(\ell^\infty, E)$ the range of T is an (∞)-space.*
(η) *For every $T \in L(\ell^\infty, E)$ and for every weakly convergent sequence $y_1, y_2, \ldots$ in ℓ^∞, the image $Ty_1, Ty_2, \ldots$ is a convergent sequence in E.*

Proof. $(\alpha) \Rightarrow (\beta)$ follows from Lemma 3.8. The implications $(\beta) \Rightarrow (\delta) \Rightarrow (\varepsilon) \Rightarrow (\alpha)$ are obvious.
Clearly $(\delta) \Rightarrow (\eta)$. Also $(\eta) \Rightarrow (\alpha)$ follows easily by applying Lemma 3.8.

LEMMA 3.10. *Let E be a locally convex space and let D be a closed subspace of countable type. If S is a subspace of countable type of E/D then $\pi^{-1}(S)$ is also of countable type, where $\pi : E \to E/D$ is the quotient map.*

Proof. Let p be a continuous seminorm on E; we produce a countable set $X \subset \pi^{-1}(S)$ such that $[X]$ is p-dense in $\pi^{-1}(S)$. There exists a countable set $F \subset S$ such that F is $\overline{p}$-dense in S where $\overline{p}$ is the quotient seminorm of p on E/D. Choose a countable set $Y \subset \pi^{-1}(S)$ with $\pi(Y) = F$ and a countable set $Z \subset D$ such that $[Z]$ is p-dense in D. Set $X := Z + Y$. To prove that $[X]$ is p-dense in $\pi^{-1}(S)$, let $x \in \pi^{-1}(S)$ and $\varepsilon > 0$. There is an $y \in [Y]$ with $\overline{p}\pi(x - y) < \varepsilon$, so there exists $d \in D$ with $p(x - y - d) < \varepsilon$. There is a $z \in [Z]$ with $p(d - z) < \varepsilon$. Then $p\big(x - (y + z)\big) < \varepsilon$ and we are done.

REMARK. By taking $S = E/D$ in the above proof we obtain the following: *If D and E/D are of countable type then so is E.*

Now we state the announced strong version of Proposition 3.7. d.ii).

PROPOSITION 3.11. *Let E be a locally convex space and let D be a closed subspace of countable type. Then, if E/D is an (∞)-space then so is E.*

Proof. Let $T \in L(\ell^\infty, E)$, let $\pi : E \to E/D$ be the quotient map. Then $\pi \circ T$ is compact so $(\pi \circ T)$ (ℓ^∞) is of countable type in E/D. By Lemma 3.10 $\pi^{-1}(\pi \circ T(\ell^\infty))$ is of countable type in E, hence so is $T(\ell^\infty)$. By Theorem 3.9 $(\gamma) \Rightarrow (\alpha)$, E is an (∞)-space.

Of similar nature is the following "3-space property" for Fréchet (i.e., complete and metrizable) spaces.

THEOREM 3.12. *Let E be a Fréchet space and let D be a closed linear subspace of E. If D and E/D are (∞)-spaces then so is E.*

Proof. Let $T \in L(\ell^\infty, E)$, let $\pi : E \to E/D$ be the quotient map. By assumption $\pi \circ T$ is compact so it has the form (Lemma 3.8)

$$(\eta_1, \eta_2, \ldots) \stackrel{\pi \circ T}{\longmapsto} \sum_{n=1}^{\infty} y_n x_n$$

where $x_n \to 0$ in E/D. By metrizability we can find a sequence $y_1, y_2, \ldots$ in E with $y_n \to 0$ and $\pi(y_n) = x_n$ for all n. By completeness, the formula

$$V(\eta_1, \eta_2, \ldots) = \sum_{n=1}^{\infty} \eta_n y_n$$

defines a compact map $V \in L(\ell^\infty, E)$. We have $\pi \circ (T - V) = 0$ so that $T - V \in L(\ell^\infty, D)$. By assumption $T - V$ is compact and so is $T = (T - V) + V$.

§4. POLARITY.

Recall ([6], Definition 3.5) that a locally convex space E is *polar* if its topology is generated by a base of polar seminorms, where a seminorm p is *polar* if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$.

It is natural to ask whether each (O.P.)-space is automatically polar. Let us show that the answer, in general, is no.

PROPOSITION 4.1. *Let (E, τ) be an (O.P.)-space admitting a non-trivial continuous seminorm p with*

$$f \in E^*, |f| \leq p \Rightarrow f = 0.$$

Then, there is a locally convex topology τ_1 with $\sigma(E, E') \subset \tau_1 \subset \tau$ such that (E, τ_1) is nonpolar but an (O.P.)-space.

Proof. We may assume that τ is polar. (otherwise take $\tau_1 := \tau$). Let τ_1 be the locally convex topology induced by $\sigma(E, E')$ and the single seminorm p. Then, clearly $\sigma(E, E') \subset \tau_1 \subset \tau$ and hence (E, τ_1) is an (O.P.)-space.

Suppose (E, τ_1) is polar; we arrive at a contradiction. There is a τ_1-continuous polar seminorm q such that $p \leq q$. As $\sigma(E, E')$ and p generate the topology τ_1 we have $q \leq \max(cp, r)$ for some $c > 0$ and some $\sigma(E, E')$-continuous seminorm r. Without loss, $r = \max(|f_1|, |f_2|, \ldots, |f_n|)$ where $f_1, \ldots, f_n \in E'$. Set $H := \bigcap_{i=1}^{n} \text{Ker}\,(f_i)$. Then on H we have $p \leq q \leq cp$. We can find a non-trivial $g \in E'$ with $|g| \leq \frac{1}{2c} q$. Then $|g| \leq \frac{1}{2} p$ on H. Since H has finite codimension we can extend g to a $\tilde{g} \in E^*$ such that $|\tilde{g}| \leq p$ on E, a contradiction.

To find an example of an (O.P.)-space which is not polar, take a set I with cardinality large enough to make possible a linear surjection $\pi : c_o(I) \to \ell^\infty / c_o$. Then $p : x \to \|\pi(x)\| \quad \big(x \in c_o(I)\big)$ satisfies the requirement of Proposition 4.1 and $(c_o(I), \tau_1)$ is the wanted space where τ_1 is the topology generated by the weak topology and the seminorm p.

The example we have above is not metrizable which is not accidental. In fact, we shall see that metrizable locally convex (O.P.)-spaces are automatically polar (see Corollary 4.4).

LEMMA 4.2. *Let E be a K-vector space and let τ_1, τ_2 be locally convex topologies both induced by countably many seminorms. Suppose τ_1-bounded $= \tau_2$-bounded for subsets of E. Then, $\tau_1 = \tau_2$.*

Proof. Let $p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots$ be a sequence of seminorms defining τ_2 and let $q_1 \leq q_2 \leq \ldots \leq q_n \leq \ldots$ be a sequence of seminorms defining τ_1. It suffices to prove that $\tau_2 \leq \tau_1$.

First we show that $p_1 \leq Cq_n$ for some constant $C > 0$ and some $n \in N$.

Suppose not. Then $p_1 \leq Cq_1$ is true for no $C > 0$ so there exists a sequence $x_{11}, x_{12}, \ldots$ in E with $q_1(x_{1n}) \leq 1$ for each n while $p_1(x_{1n}) \geq n$ for each n. Also $p_1 \leq Cq_2$ is true for no $C > 0$ so there exists a sequence $x_{21}, x_{22}, \ldots$ in E with $q_2(x_{2n}) \leq 1$ and $p_1(x_{2n}) \geq n$ for each n, etc.
Inductively we find a double sequence

$$\begin{array}{ccc} x_{11}, & x_{12}, & x_{13}, \ldots \\ x_{21}, & x_{22}, & x_{23}, \ldots \\ \vdots & \vdots & \vdots \end{array}$$

in E such that $q_k(x_{kn}) \leq 1$ for each k, n while $p_1(x_{kn}) \geq n$ for each k, n. We see that the diagonal sequence $x_{11}, x_{22}, \ldots$ is q_k-bounded for each k so it is τ_1-bounded. But $p_1(x_{nn}) \geq n$, which implies that the sequence $x_{11}, x_{22}, \ldots$ is not τ_2-bounded: a contradiction.

In a similar way as above we can prove that $p_2, p_3, \ldots$ are majorized by positive multiples of some q_n. It follows that $\tau_2 \leq \tau_1$.

THEOREM 4.3. *Let (E, τ) be a metrizable locally convex space such that every weakly bounded set is bounded. Then E is polar.*

Proof. Let $p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots$ be an increasing sequence of seminorms inducing τ. Let, for each n, $\tilde{p}_n$ be the polar seminorm associated to p_n (i.e. $\tilde{p}_n = \sup\{|f| : f \in E^*, |f| \leq p_n\}$). We have $\tilde{p}_1 \leq \tilde{p}_2 \leq \ldots$, the topology induced by these $\tilde{p}_n$ is polar. It is easily seen that $\tilde{\tau}$ is the strongest polar topology that is $\subseteq \tau$. Now let $X \subset E$. By [6] Theorem 7.5 we have X is weakly bounded $\iff$ X is $\tilde{\tau}$-bounded. Then, by assumption, τ-bounded $=\tilde{\tau}$-bounded. Then $\tau = \tilde{\tau}$ by Lemma 4.2, i.e., τ is polar.

COROLLARY 4.4. *A metrizable* (O.P.)*-space is polar.*

Proof. It is a direct consequence of Corollary 2.4 i) $\Rightarrow$ ii) and Theorem 4.3.

REMARKS.

1. There are metrizable (C.O.P.)-spaces which are not polar.
   Example: Take $E = \bigoplus_n \ell_n^\infty$ as in [13] Exercise 4.5. E is a Banach space whose dual E' separates the points of E and which is not polar.

On the other hand, by looking at the proof of Theorem 2.2 it is very easy to see that a Banach space is a (C.O.P.)-space if and only if its dual separates the points. Then $E = \bigoplus_n \ell_n^\infty$ as above is a (C.O.P.)-space which is not polar.

2. As we know (see p.5), every (B.O.P.) normed space E is an (O.P.)-space and hence it is polar.

   Corollary 4.4 allows us to formulate the following question:

PROBLEM. E is a metrizable (B.O.P.)-space $\Rightarrow$ E is polar?

Now, we shall prove interesting characterizations of *polar* (∞)-spaces. The heart of the matter is contained in the next lemma.

LEMMA 4.5. *Let (E,τ) be a polar locally convex space. Suppose E does not contain (a subspace linearly homeomorphic to) ℓ^∞. Then E is an (∞)-space.*

Proof. Let $T \in L(\ell^\infty, E)$ be not compact; we derive a contradiction by showing that ℓ^∞ is a subspace of E. Let $e_1, e_2, \ldots$ be the unit vectors of ℓ^∞. Then $\{Te_1, Te_2, \ldots\}$ is not a compactoid (otherwise, TB_{ℓ^∞}, being a subset of the weak closure of $co\{Te_1, Te_2, \ldots\}$ would be a compactoid, so T would be compact). So, there exists a continuous polar seminorm p such that $\{Te_1, Te_2, \ldots\}$ is not p-compactoid. By [11], Theorem 2 there exists a $t \in (0,1)$ and a subsequence $z_1, z_2, \ldots$ of $Te_1, Te_2, \ldots$ that is t-orthogonal with respect to p and such that $\inf_n p(z_n) > 0$. Without loss, assume $p(z_n) \geq 1$ for each n.

Now, inductively we shall construct a subsequence $u_1, u_2, \ldots$ of $z_1, z_2, \ldots$ and $f_1, f_2, \ldots \in E'$ such that $|f_n| \leq 2t^{-1}p$ for all n and

$$|f_m(u_n)| = \begin{cases} 0 & \text{if } m > n \\ 1 & \text{if } m = n \end{cases}, \; |f_m(u_n)| \leq \frac{1}{2} \text{ if } m < n$$

To do that, observe that the function $h_1 : \lambda z_1 \mapsto \lambda \quad (\lambda \in K)$ satisfies $|h_1| \leq p$. By polarity it can be extended to an $f_1 \in E'$ such that $|f_1| \leq 2p$. Set $u_1 := z_1$. Suppose $f_1, \ldots, f_{m-1}$ and $u_1, \ldots, u_{m-1}$ are chosen with the required properties. Since $Te_n \to 0$ weakly we have $z_n \to 0$ weakly. So we can find a k (larger than the indexes with respect to z of $u_1, \ldots, u_{m-1}$) such that $|f_1(z_n)| \leq 1/2, \ldots, |f_{m-1}(z_n)| \leq 1/2$ for $n \geq k$. Choose $u_m := z_k$. The function $h_m : \lambda_1 u_1 + \cdots + \lambda_m u_m \mapsto \lambda_m \quad (\lambda_1, \ldots, \lambda_m \in K)$ satisfies $|h_m| \leq t^{-1}p$ so it can be extended to a function $f_m \in E'$ such that $|f_m| \leq 2t^{-1}p$. We see that $f_1, \ldots, f_m$ and $u_1, \ldots, u_m$ have the required properties.

Now, we have that $u_1, u_2, \ldots$ is a subsequence, say $Te_{i_1}, Te_{i_2}, \ldots$ of $Te_1, Te_2, \ldots$. Define a linear isometry $\Omega : \ell^\infty \to \ell^\infty$ by the formula

$$\big(\Omega(y_1, y_2, \ldots)\big)_n = \begin{cases} 0 & \text{if } n \notin \{i_1, i_2, \ldots\} \\ y_n & \text{if } n \in \{i_1, i_2, \ldots\} \end{cases}$$

and set $S := T \circ \Omega$. Then obviously $S \in L(\ell^\infty, E)$ and S is described by the formula

$$S(y_1, y_2, \ldots) = \sigma(E, E') - \sum_{n=1}^{\infty} y_n u_n.$$

Finally let $y = (y_1, y_2, \ldots) \in \ell^\infty$, $y \neq 0$. There is an $m \in N$ such that $|y_m| > \frac{1}{2}\|y\|$. We have $p(Sy) \geq \frac{1}{2}t|f_m(Sy)| = \frac{1}{2}t|\sum_{n\geq m} y_n f_m(u_n)|$. If $n > m$ we have $|y_n f_m(u_n)| \leq \frac{1}{2}|y_n| \leq \frac{1}{2}\|y\|$ whereas $|y_m f_m(v_m)| = |y_m| > \frac{1}{2}\|y\|$ so $p(Sy) \geq \frac{1}{2}\|y\|$ implying that S is a linear homeomorphism from ℓ^∞ onto $S(\ell^\infty) \subset E$ which gives the desired contradiction.

LEMMA 4.6. *Let E be a polar locally convex space and let $i : \ell^\infty \to E$ be a linear homeomorphism of ℓ^∞ onto $i(\ell^\infty)$. Then there exists a continuous linear map $P : E \to \ell^\infty$ such that $P \circ i$ is the identity on ℓ^∞.*

Proof. There exists a continuous polar seminorm p on E such that $x \mapsto p(i(x))$ $(x \in \ell^\infty)$ is equivalent to the standard norm on ℓ^∞. Let $\overline{p} : E/\mathrm{Ker}p \to [0, \infty)$ be the quotient norm of p and let $\pi : E \to E/\mathrm{Ker}p$ be the quotient map. The map

$$\pi \circ i : \ell^\infty \to E \to E/\mathrm{Ker}p$$

is a linear homeomorphism of ℓ^∞ into the normed space $(E/\mathrm{Ker}p, \overline{p})$. Then (see [10]) there exists a linear continuous map $Q : (E/\mathrm{Ker}\overline{p}, \overline{p}) \to \ell^\infty$ such that $Q \circ \pi \circ i$ is the identity on ℓ^∞. Now set $P := Q \circ \pi$.

COROLLARY 4.7. *For a polar locally convex space (E, τ) the following are equivalent.*

(a) *E is an (∞)-space.*
(b) *For every $T \in L(\ell^\infty, E)$, $Te_n \to 0$ in E (where $e_1, e_2, \ldots$ are the unit vectors of ℓ^∞).*
(c) *For every $T \in L(\ell^\infty, E)$ the restriction $T|c_o$ is compact.*
(d) *E does not contain a subspace linearly homeomorphic to ℓ^∞.*
(e) *E does not contain a complemented subspace linearly homeomorphic to ℓ^∞.*
If in addition E is weakly sequentally complete, properties $(a) - (e)$ are equivalent to
(f) *E is an (O.P.)-space.*
(g) *Every bounded absolutely convex and $\sigma(E, E')$-metrizable subset of E is compactoid.*
(h) *If F is a locally convex space and $T \in L(F, E)$ then T maps weakly convergent sequences in F into convergent sequences in E.*

Proof.

(a) $\Rightarrow$ (b) follows by Theorem 3.9 (α) $\Rightarrow$ (η).

(b) $\Rightarrow$ (c). Let $T \in L(\ell^\infty, E)$. Given $x = \sum_{n=1}^{\infty} x_n e_n$ in c_o we have $Tx = \sum_{n=1}^{\infty} x_n T e_n$. So T maps the unit ball of c_o into $\overline{co}\{Te_1, Te_2, \ldots\}$ which is a compactoid set since $Te_n \to 0$.

(c) $\Rightarrow$ (d) is obvious; (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (a) follow from Lemmas 4.5 and 4.6.

Now assume that E is weakly sequentally complete.

The equivalence (a) $\iff$ (f) is a direct consequence of Theorem 3.2 (α) $\iff$ (β).

The implication (f) $\Rightarrow$ (g) follows from Corollary 2.4 i) $\Rightarrow$ iii).

(g) $\Rightarrow$ (h). Let F be a locally convex space and let $T \in L(F, E)$. If $x_n \to 0$ weakly in F clearly we have $Tx_n \to 0$ weakly in E, so ([8], Theorem 6.1) $\overline{co}^{\sigma(E,E')}\{Tx_1, Tx_2, \ldots\}$ is $\sigma(E, E')$-metrizable and by (g) it is a compactoid set. Thus, $Tx_n \to 0$ in τ.

Finally, observe that by taking $F = \ell^\infty$ in property (h) we derive property (η) of Theorem 3.9 and hence E is an (∞)-space.

§5. METRIZABLE (O.P.)-SPACES.

PROPOSITION 5.1. *Let (E, τ) be a metrizable locally convex space and let D be a dense subspace of E. If D is an* (O.P.)*-space then so is E.*

Proof. There is an invariant metric d on E inducing τ. Let $x_1, x_2, \ldots$ be a sequence in E such that $\lim_{n\to\infty} x_n = 0$ in $\sigma(E, E')$. For each n, choose a $y_n \in D$ with $d(x_n, y_n) \leq \frac{1}{n}$. Then $x_n - y_n \xrightarrow{\tau} 0$ so $x_n - y_n \to 0$ in $\sigma(E, E')$ and also $y_n = y_n - x_n + x_n \to 0$ in $\sigma(E, E')$ hence in $\sigma(D, D')$. Since D is an (O.P.)-space, $y_n \xrightarrow{\tau} 0$. But then $x_n = x_n - y_n + y_n \xrightarrow{\tau} 0$.

PROBLEM. Does the conclusion hold if we drop the metrizability condition?

A standard application of the Closed Graph Theorem yields the following lemma.

LEMMA 5.2. *Let $(E, \tau_1), (F, \tau_2)$ be Fréchet spaces and let $T : E \to F$ be a linear map. Suppose $(F, \tau_2)'$ separates the points of F and that $T : (E, \tau_1) \to \big(F, \sigma(F, F')\big)$ is continuous. Then $T : (E, \tau_1) \to (F, \tau_2)$ is continuous.*

The following results give some characterizations of metrizable (O.P.)-spaces.

THEOREM 5.3. *For a Fréchet space E the following are equivalent:*
(α) E is an (O.P.)*-space.*
(β) E' separates the points of E, E is weakly sequentally complete, E is an (∞)-space.

Proof.

$(\alpha) \Rightarrow (\beta)$. That E is an (∞)-space with separating dual follows from Theorem 3.2 $(\alpha) \Rightarrow (\beta)$.
Also if $x_1, x_2, \ldots$ is weakly Cauchy then $x_{n+1} - x_n \to 0$ weakly hence strongly. As E is Fréchet, $x_n \to x$ strongly for some $x \in E$, hence weakly. Then, E is weakly sequentally complete.

$(\beta) \Rightarrow (\alpha)$. Let $x_1, x_2, \ldots$ be a sequence in E tending weakly to 0. Then the formula

$$(\eta_1, \eta_2, \ldots) \xmapsto{T} \sigma(E, E') - \sum_{i=1}^{\infty} \eta_i x_i$$

defines, by weakly sequential completeness a linear map $T : \ell^\infty \to E$. It is easily seen that T is strong to weak continuous. By Lemma 5.2, T is continuous. By Theorem 3.9 $(\alpha) \iff (\eta)$ we conclude that $x_n \to 0$ in the initial topology of E.

THEOREM 5.4. *For a metrizable locally convex space E the following are equivalent.*
(α) *E is a complete* (O.P.)*-space.*
(β) *E is a polar weakly sequentally complete* (∞)*-space.*

Proof. $(\alpha) \Rightarrow (\beta)$ follows from Theorem 5.3 and Corollary 4.4. To prove $(\beta) \Rightarrow (\alpha)$ observe that from Theorem 3.2 $(\beta) \Rightarrow (\alpha)$ it follows that E is an (O.P.)-space. To prove completeness, let $x_1, x_2, \ldots$ be a Cauchy sequence in E. It is weakly Cauchy so that $x_n \to x$ weakly for some $x \in E$. Since E is an (O.P.)-space we conclude that $x_n \to x$ in the initial topology of E.

For metrizable (∞)-spaces we have the following extension of Corollary 4.7.

PROPOSITION 5.5. *For a polar Fréchet space the following are equivalent.*
(α) *E is an (∞)-space.*
(β) *No continuous linear map $\ell^\infty \to E$ is semi-Fredholm* (A continuous linear map T is *semi Fredholm* if its Kernel, $\mathrm{Ker}(T)$ is finite dimensional and its range, $T\ell^\infty$, is closed).
(γ) *For every $T \in L(\ell^\infty, E)$ there exists a compact map $S \in L(\ell^\infty, E)$ such that* $\mathrm{Ker}(T - S)$ *is infinite dimensional.*

Proof.

$(\alpha) \Rightarrow (\beta)$. Assume there exists a semi-Fredholm $T : \ell^\infty \to E$. Then the corresponding bijection $\ell^\infty/\mathrm{Ker}(T) \to T\ell^\infty$ is a linear isomorphism ([5], Corollary 2.75). Compactness of T implies that the canonical quotient map $\ell^\infty \to \ell^\infty/\mathrm{Ker}(T)$ is compact, a contradiction.

$(\beta) \Rightarrow (\alpha)$ follows directly from Corollary 4.7 (d) $\Rightarrow$ (a).

$(\alpha) \Rightarrow (\gamma)$ is obvious. (Choose $S := T$.)

$(\gamma) \Rightarrow (\alpha)$ is also a consequence of Corollary 4.7 (d) $\Rightarrow$ (a): Observe that if there is an injection $i : \ell^\infty \to E$ such that $i(\ell^\infty)$ is isomorphic to ℓ^∞, then by (γ) there is a compact map $S \in L(\ell^\infty, E)$ such that $\mathrm{Ker}(i - S)$ is infinite dimensional. Since $i \mid \mathrm{Ker}(i - S) = S \mid \mathrm{Ker}(i - S)$ we derive that the restriction $i \mid \mathrm{Ker}(i - S)$ is a compact map, which is impossible (see [13], Theorem 4.40).

§6. BANACH (O.P.)-SPACES.

The following definition is in a sense dual to the definition of (∞)-space (see 3.1).

DEFINITION 6.1. A locally convex space E is said to be a (0)-*space* if every continuous linear map $T : E \to c_o$ is compact (i.e., there exists a continuous seminorm p on E such that $T\{x \in E : p(x) \leq 1\}$ is a compactoid in c_o).

In this section we study, for Banach spaces, the duality between (0)-spaces on one hand and (∞)-spaces or (O.P.)-spaces on the other.

(0)-SPACES have been studied by N. DE GRANDE - DE KIMPE in [1] and [2] (here the base field was spherically complete) and by T. KIYOSAWA in [4]. Putting together Theorem 8 of [2] (which also works for non-spherically complete fields) and Theorem 14 of [4] we obtain the following characterizations of Banach (0)-spaces.

THEOREM 6.2. (see [2] and [4]). *For a Banach space E the following are equivalent.*

(α) *E is a (0)-space.*

(β) *E does not contain a complemented subspace linearly homeomorphic to c_o* (Recall that every infinite dimensional Banach space contains a subspace which is isomorphic to c_o).

(γ) *No quotient of E is isomorphic to c_o.*

(δ) *In E' is every $\sigma(E', E)$-convergent sequence also norm convergent.*

(ε) *Let (T_n) be a sequence of compact continuous linear maps from E to a Banach space F, converging pointwise to T. Then T is compact.*

(η) *The space $C(E, c_o)$ of compact continuous linear maps from E to c_o is complemented in $L(E, c_o)$.*

From this result we derive

COROLLARY 6.3. *For a polar Banach space E we have the following.*

(i) *E is a (0)-space $\iff$ E' is an (O.P.)-space $\iff$ E' is an (∞)-space.*

(ii) *E' is a (0)-space $\Rightarrow$ E is an (∞)-space.*

If in addition there exists a closed subspace D of E'' (the bidual of E) such that D is an (∞)-space and E''/D is isomorphic to E (e.g., *when E is reflexive), then*

(iii) *E is an (∞)-space $\Rightarrow$ E' is a (0)-space.*

Proof.

(i) Assume E is a (0)-space and let $f_1, f_2, \dots$ be a sequence in E' such that $f_n \to 0$ in $\sigma(E', E'')$. Then $f_n \to 0$ in $\sigma(E', E)$ and from Theorem 6.2 $(\alpha) \Rightarrow (\delta)$ we obtain that $f_1, f_2, \dots$ is norm convergent in E', i.e. E' is an (O.P.)-space.

Clearly, if E' is an (O.P.)-space then E' is an (∞)-space (see Theorem 3.2).

Now assume E' is an (∞)-space and let $T \in L(E, c_o)$. Since $T' \in L(\ell^\infty, E')$ is compact we derive that T is also compact ([9], Proposition 5.8), i.e. E is a (0)-space.

Property (ii) follows directly from the definition of (∞)-and (0)-spaces and from [9], Proposition 5.8.

(iii) Theorem 3.12 implies that under the assumptions of (iii), E'' is an (∞)-space. Now apply (i) to conclude that E' is a (0)-space.

PROBLEM. Let E be a Banach space. If E is an (∞)-space, does it imply that E' is a (0)-space?

REMARKS.

1. Let I be a small set (i.e. the cardinality of I is nonmeasurable). By [13] Theorem 4.21, $c_o(I)$ is a reflexive Banach space and by Theorem 1.6 vi) it is also a (∞)-space. Applying Corollary 6.3 (iii) we deduce that $\ell^\infty(I)$ is a (0)-space. Also we have (we like to thank Arnoud van Rooij for the proof)

PROPOSITION 6.4. *If K is small then for every index set I, $\ell^\infty(I)$ is a (0)-space.*

Proof.

Let $T \in L(\ell^\infty(I), c_o)$. Since K is small, c_o is also small. So, there exists a small set $A \subset \ell^\infty(I)$ such that $TA = T\ell^\infty(I)$.

Define in I the following equivalence relation

$$i_1 \sim i_2 \iff a(i_1) = a(i_2) \quad \forall a \in A$$

and let $[i]$ denote the class of $i \in I$. Let J be the collection of these classes. Let $\pi : I \to J$ be the canonical surjection. Then π induces a continuous linear map $P : \ell^\infty(J) \to \ell^\infty(I)$. It is easily seen that $A \subset P\ell^\infty(J)$.

There is also an injection $J \xrightarrow{\varphi} K^A$ given by:
If $[i] \in J$, then $\varphi([i])$ is the map $a \mapsto a(i)$.
Then J is also small.

Thus by the above remark, $T \circ P : \ell^\infty(J) \to c_o$ is compact. But $(T \circ P)(\ell^\infty(J)) \supset T(A) = T\ell^\infty(I)$. Hence, T is compact ([13], Theorem 4.40).

2. The "duals" of the properties $(\alpha)-(\delta)$ in Theorem 6.2 have been studied in this paper (see Corollaries 4.7 and 6.3). Property (ε) has also a counterpart. In fact, we have

PROPOSITION 6.5. *Let E be a weakly sequentially complete Banach space such that E' separates the points of E. Then the following are equivalent.*

(α) *E is an (O.P.)-space.*

(β) *Let F be a Banach (0)-space and let (T_n) be a sequence of compact continuous linear maps from F to E such that $T_n(y) \to T(y)$ in $\sigma(E, E')$ for each $y \in F$. Then T is compact.*

Proof.

$(\alpha) \Rightarrow (\beta)$. Let F and (T_n) be as in (β). By (O.P.) we have that $T_n(y) \to T(y)$ in norm for each $y \in F$. Now apply Theorem 6.2 $(\alpha) \Rightarrow (\varepsilon)$ to conclude that T is compact.

$(\beta) \Rightarrow (\alpha)$. By Theorem 5.3 it suffices to prove that E is an (∞)-space. So let $T \in L(\ell^\infty, E)$ and for each $m \in N$ let $T_m : \ell^\infty \to E$ be given by

$$(\eta_1, \eta_2, \ldots)\backslash \longmapsto \sum_{n=1}^{m} \eta_n T e_n.$$

It follows easily that $T_m(x) \to T(x)$ in $\sigma(E, E')$ for each $x \in \ell^\infty$. Also, every T_m is a continuous linear map of finite rank, so it is compact. By (β) we conclude that T is compact.

Observe that in property (β) of Proposition 6.5 the condition of F being a (0)-space cannot be dropped. In fact, take $E = F = c_o$ and let $T_n : c_o \to c_o$ be given by

$$T_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n, 0, 0, \ldots) \quad (n \in \mathbb{N}).$$

Then (T_n) is a sequence of finite rank (and hence compact) continuous linear maps converging pointwise to the identity map on c_o, which is not compact.

3. By considering the "dual counterpart" of property (η) in Theorem 6.2, we obtain - for Banach spaces - the following improvement of Theorem 3.2.

PROPOSITION 6.6. *For a Banach space E we consider the following properties:*

(α) *E is an (O.P.)-space.*

(β) *The space $C(\ell^\infty, E)$ of all compact continuous linear maps from ℓ^∞ to E is complemented in $L(\ell^\infty, E)$ and every weakly bounded set in E is bounded.*

Then $(\alpha) \Rightarrow (\beta)$. If, in addition, E is weakly sequentially complete we have also $(\beta) \Rightarrow (\alpha)$.

Proof.

$(\alpha) \Rightarrow (\beta)$ follows from Theorem 3.2.

$(\beta) \Rightarrow (\alpha)$ can be proved similarly to Theorem 14 of [4]. In fact, observe that if E is not an (O.P.)-space there exists a sequence (x_n) in E such that $x_n \to 0$ in $\sigma(E, E')$ and $1 \leq \|x_n\| \leq 1/|\pi|$ $\forall n$ (where $\pi \in K, |\pi| > 1$ is fixed). For $\lambda = (\lambda_n) \in \ell^\infty$ we define $H_\lambda \in L(\ell^\infty, E)$ by $H_\lambda(\alpha_1, \alpha_2, \ldots) = \sigma(E, E') - \sum_{n=1}^{\infty} \alpha_n \lambda_n x_n \quad ((\alpha_1, \alpha_2, \ldots) \in \ell^\infty)$. From now we can follow the proof as in Theorem 14 of [4].

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Address of first author: Department of Mathematics, Facultad de Ciencias, Universidad de Cantabria, Avda de los Castros s/n, Santander, Spain.

Address of second author: See cover.